The two-body problem

One of the classic problems of mechanics is the motion of two mass points M_1 and M_2 subject only to a conservative force derived from a potential that depends only on the distance between the points. We will completely solve this problem using the Lagrangian formalism.

There are no constraints on the Cartesian coordinates of the masses, $\vec{r}_1$ and $\vec{r}_2$, so we write the Lagrangian in terms of them:

$$L = \frac{1}{2}M_1|\dot{\vec{r}}_1|^2 + \frac{1}{2}M_2|\dot{\vec{r}}_2|^2 - V(|\vec{r}_1 - \vec{r}_2|)$$

We will keep the functional form of the potential general for now; later we will specialize to gravitational and electro-static potentials.

The structure of L simplifies when we make a linear change of variables to a new set of generalized coordinates. From experience we know that the motion of the system center-of-mass is very simple for an isolated system, so that will be one of our new coordinates:

$$\vec{R} = \frac{M_1 \vec{r}_1 + M_2 \vec{r}_2}{M_1 + M_2} .$$

And since the potential energy depends only on the difference of the mass positions, we will use that as the other combination:

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

Expressing $\vec{r}_1$ and $\vec{r}_2$ (which appear individually in the kinetic energy) in terms of $\vec{R}$ and $\vec{r}$ is a routine exercise in algebra:

$$\vec{R} + \frac{M_2}{M_1+M_2}\vec{r} = \vec{r}_1$$

$$\vec{R} - \frac{M_1}{M_1+M_2}\vec{r} = \vec{r}_2$$

Here's what we obtain for the kinetic energy:

$$T = \tfrac{1}{2}M_1\left|\dot{\vec{R}} + \frac{M_2}{M_1+M_2}\dot{\vec{r}}\right|^2 + \tfrac{1}{2}M_2\left|\dot{\vec{R}} - \frac{M_1}{M_1+M_2}\dot{\vec{r}}\right|^2$$

$$= \tfrac{1}{2}(M_1+M_2)\left|\dot{\vec{R}}\right|^2 +$$

$$\frac{1}{2}\underbrace{\frac{M_1M_2^2 + M_2M_1^2}{(M_1+M_2)^2}}_{\frac{M_1M_2}{M_1+M_2} = \mu}\left|\dot{\vec{r}}\right|^2$$

The combination of masses that appears in the relative position term is called the "reduced mass", μ.

Here is the Lagrangian in the new coordinates:

$$L = \tfrac{1}{2}(M_1+M_2)|\dot{\vec{R}}|^2 + \tfrac{1}{2}\mu|\dot{\vec{r}}|^2 - V(|\vec{r}|)$$

Since $\vec{R}$ does not appear in L, its conjugate momentum is conserved:

$$\vec{P}_R = \frac{\partial L}{\partial \dot{\vec{R}}} = (M_1+M_2)\dot{\vec{R}}$$

This is just the total momentum of the system (as defined in freshman mechanics) which ~~the~~ in the absence of external forces is constant in time (conserved).

We use the Euler-Lagrange equations to find the equation of motion for the relative position, $\vec{r}$:

$$\frac{\partial L}{\partial \vec{r}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}}\right)$$

$$-V'(|\vec{r}|)\underbrace{\frac{\partial |\vec{r}|}{\partial \vec{r}}}_{\hat{r}} = \mu \ddot{\vec{r}}$$

$V'(\cdots)$ = derivative of the function V with respect to its argument

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equation of motion for $\vec{r}$:

$$\mu \ddot{\vec{r}} = - V'(|\vec{r}|)\, \hat{r}$$

This is the classical 1-body problem, of a point mass μ subject to a <u>central</u> <u>force</u>

$$\vec{F} = - V'(|\vec{r}|)\, \hat{r} .$$

The vector $\vec{r}$ and its velocity $\vec{v} = \dot{\vec{r}}$ define a plane with normal vector $\vec{v} \times \vec{r}$. It's easy to check that this normal vector, and hence the plane defined by $\vec{v}$ and $\vec{r}$, is constant in time:

$$\frac{d}{dt}(\dot{\vec{r}} \times \vec{r}) = \ddot{\vec{r}} \times \vec{r} = 0,$$

Since $\ddot{\vec{r}}$ is parallel to $\vec{r}$ by the equations of motion. So if $\vec{v}$ and $\vec{r}$ stay in the same plane for all times, we let $\vec{r}$ from now on be a vector in the x-y plane.

Return to the Lagrangian

$$L = \text{const.} + \tfrac{1}{2}\mu|\dot{\vec{r}}|^2 - V(|\vec{r}|)$$

where const. = (kinetic energy of center of mass motion) and rewrite in terms of polar coordinates:

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$L = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

(We neglect the constant CM kinetic energy from now on.)

In our 2 DoF Lagrangian the coordinate θ is absent, so its conjugate momentum is conserved:

$$L_z = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{constant}$$

We use the notation "L_z" instead of p_θ because this conjugate momentum is the familiar "angular momentum" of freshman mechanics.

We have succeeded in reducing the original 2-body problem, with 6 degrees of freedom, to the motion of a single degree of freedom, r:

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)$$

$$\Downarrow$$

$$\underbrace{\mu r \dot{\theta}^2}_{\frac{L_z^2}{\mu r^3}} - V'(r) = \mu \ddot{r}$$

$$\mu \ddot{r} = \frac{L_z^2}{\mu r^3} - V'(r)$$

We will study this equation in detail, eventually establishing the elliptical shape of planetary orbits, as well as other things.